Half Logistic Exponentiated Inverted Weibull distribution: Properties and Application to Survival data

[1]Oseghale Osezuwa Innocent, [2]Laoye Victoria Eshomomoh, [3]Adeleke Zainab Adekemi
[1]Department of Mathematics and Statistics, Federal University Otuoke, Bayelsa State.
[2,3]Department of Statistics, University of Ibadan, Oyo State.
Corresponding author: innocentoseghale@gmail.com
DOI: 10.56201/ijasmt.vol.11.no2.2025.pg100-110

Abstract
In this work, a three-parameter extension of the Inverted Weibull distribution, called the half-logistic Exponentiated Inverted Weibull distribution, is proposed. This distribution has sub-models, including the Exponentiated Inverted Weibull and Inverted Weibull distributions. The proposed model is quite flexible and adaptable to model any kind of life-time data. Standard statistical properties, such as ordinary and incomplete moments, quantile function, moment generating function, reliability function, Renyi, and $\boldsymbol{\delta}$*-entropy are obtained. The maximum likelihood method is used to obtain the estimate of the model parameters. An application of the developed model to pig data illustrates its flexibility for lifetime data modeling.*

Keywords: *Maximum likelihood estimation, incomplete moments, δ-entropy, half-logistic Exponentiated Inverted Weibull distribution.*

1.0 Introduction

The inverted Weibull distribution is a popular probability distribution used in the analysis of life time data exhibiting monotone failure rates. Khan et al. (2008) studied the flexibility and tractability of the three parameters inverted Weibull (IW) distribution. Flair et al. (2012) investigated the properties of the Exponentiated inverted Weibull (EIW) distribution and its application to failure data. Mudholka (1995) developed the exponentiated Weibull distribution and applied the new distribution to model the bus-motor failure time data. Mudholkar and Hutson (1995) reviewed the exponentiated Weibull distribution with new measures. The Weibull inverted Weibull distribution was studied by Ogunde et al. (2018), modified extended Weibull distribution by Ogunde et al. (2022). We say that the random variable X has a standard EIW if its cumulative distribution function (CDF) is of the form:

$$G(x;\alpha,\theta) = 1 - \left(e^{-x^{-\theta}}\right)^{\alpha}, \tag{1}$$

And the corresponding density function (PDF) is given by

$$g(x;\alpha,\theta) = \alpha x^{-\theta-1}\left(e^{-x^{-\theta}}\right)^{\alpha}, \tag{2}$$

Where α and $\theta$are positive shape parameters.

1.1 Motivation of study

The main purpose of the modification and extension forms of the Inverted Weibull distribution is to describe and fit the data sets with non-monotonic hazard rate, such as the bathtub, unimodal and modified unimodal hazard rate. Many modifications of the Inverted Weibull distribution have achieved the above purpose. On the other hand, unfortunately, the number of parameters has increased, the forms of the survival and hazard functions have been complicated and estimation problems have risen. This work presents a more flexible representation of the extended Inverted Weibull distribution which can be used to model data exhibiting various shapes of the hazard function.

2.0 Half Logistic Exponentiated Inverted Weibull distribution

Based on the work of Cordeiro et al. (2016), the CDF of half-logistic generalized family is defined by:

$$F(x;\rho,\mathcal{N}) = \frac{1-[1-G(x;\mathcal{N})]^{\rho}}{1+[1-G(x;\mathcal{N})]^{\rho}}, \quad x \in R \tag{3}$$

where ρ is a positive shape parameter and $F(x;£)$ represents a CDF of a continuous distribution. Here, $\mathcal{N}$ represents a vector of parameter(s) related to the corresponding standard probability or baseline distribution. The chief motivations behind the half-logistic family of distributions are as follows. Cordeiro et al. (2016) demonstrated that the effect of the power transformation can enrich the baseline distribution, improving the flexibility provides a positive impact on the analyses of lifetime data sets. The normal, Fréchet, Weibull, and inverse Lomax distributions have been used as baseline in the previous work using the half-logistic generated family by Cordeiro et al. (2016). In recent studies, Ogunde et al. (2017) developed and studied the type 1 half logistic Gompertz distribution, Anwar and Bibi (2018) studied the Half-Logistic Generalized Weibull Distribution, type I half-logistic Topp-Leone distribution was proposed and studied by Zein-Eldin et al. (2019). Type I half logistic power Lomax and half logistics generalized Rayleigh were respectively studied by distribution was developed and studied by Fayomi (2019) and Ogunde et al. (2024). However, the half logistic-g family has not been completely explored and, based on the studies carried out in the past, more work needs to be done to fully explore the richness of this generated family of distribution. In this paper, we consider the baseline distribution as Inverted Weibull (IW) distribution.

Thus, we introduce a new flexible lifetime distribution with three parameters called the half-logistic Exponentiated Inverted Weibull ($HLEIW$) distribution. The CDF of the $HLEIW$ distribution with parameter vector $\zeta = (\eta, \varphi, \rho)$ is obtained by inserting (1) into (2) as

$$F(x;\alpha,\theta,\rho) = \frac{1-\left[1-\left(e^{-x^{-\theta}}\right)^{\alpha}\right]^{\rho}}{1+\left[1-\left(e^{-x^{-\theta}}\right)^{\alpha}\right]^{\rho}}, \quad x \in R \tag{4}$$

The associated PDF to (4) is given by

$$f(x;\alpha,\theta,\rho) = \frac{2\alpha\theta\rho x^{-\theta-1}\left(e^{-x^{-\theta}}\right)^{\alpha}\left[1-\left(e^{-x^{-\theta}}\right)^{\alpha}\right]^{\rho-1}}{\left(1+\left[1-\left(e^{-x^{-\theta}}\right)^{\alpha}\right]^{\rho}\right)^{2}}. \quad x \in R \tag{5}$$

Where α, θ and ρ are positive shape parameters. The survival $(S(x))$, hazard $(h(x))$,and the cumulative hazard $(H(x))$ are, respectively, given as

$$S(x;\alpha,\theta,\rho) = 1 - F(x;\alpha,\theta,\rho) = \frac{2\left[1-\left(e^{-x^{-\theta}}\right)^{\alpha}\right]^{\rho}}{1+\left[1-\left(e^{-x^{-\theta}}\right)^{\alpha}\right]^{\rho}}, \qquad x \in R \tag{6}$$

and $S(x;\alpha,\theta,\rho) = 1$ for $x \leq 0$,

$$h(x;\zeta) = \frac{g(x;\zeta)}{S(x;\zeta)} = \frac{2\alpha\theta\rho x^{-\theta-1}e^{-x^{-\theta}}\left(e^{-x^{-\theta}}\right)^{\alpha-1}}{\left(1+\left[1-\left(e^{-x^{-\theta}}\right)^{\alpha}\right]^{\rho}\right)\left(1-\left(e^{-x^{-\theta}}\right)^{\alpha}\right)}, \tag{7}$$

and $h(x;\zeta) = 0$ for $x \leq 0$, and

$$H(x) = -log[S(x;\zeta)]$$

$$= -\log(2) - \rho log\left[1-\left(e^{-x^{-\theta}}\right)^{\alpha})\right] + log\left(1+\left[1-\left(e^{-x^{-\theta}}\right)^{\alpha}\right]^{\rho}\right). \tag{8}$$

and $H(x;\zeta) = 0$ for $x \leq 0$. The graph of distribution, density, survival function(sf) and hazard function (hf) are respectively, given in Figures 1and 2 for various hypothetical values of the parameters of the distributions.

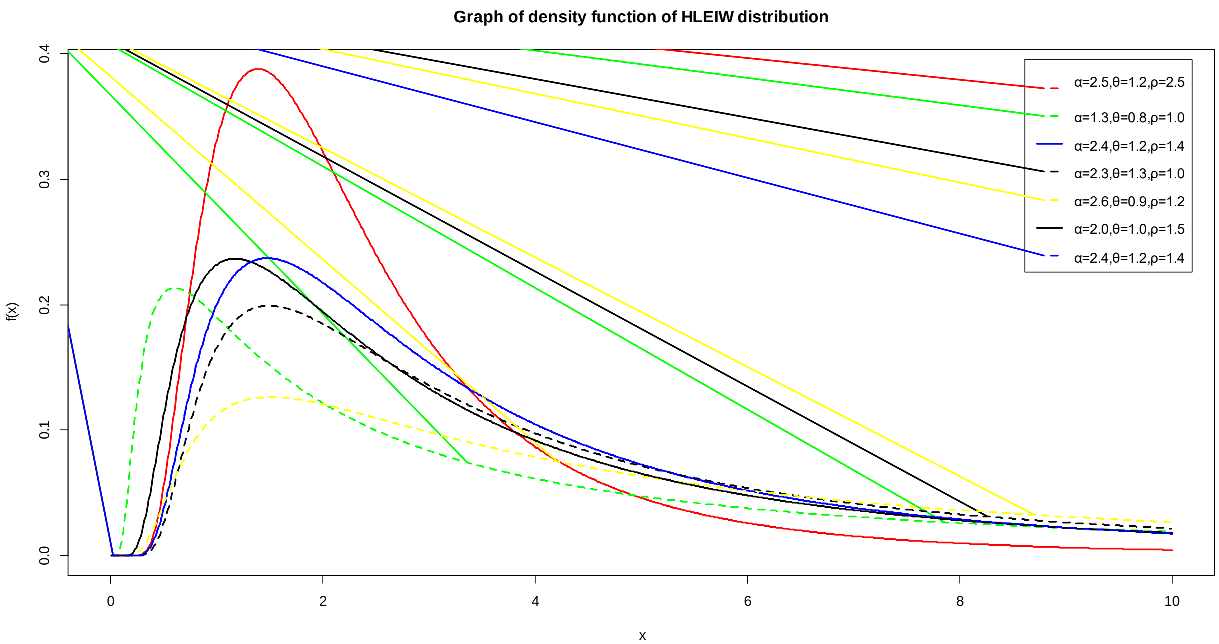

Figure 1.0. Graph of the density function of the $HLEIW$ distribution

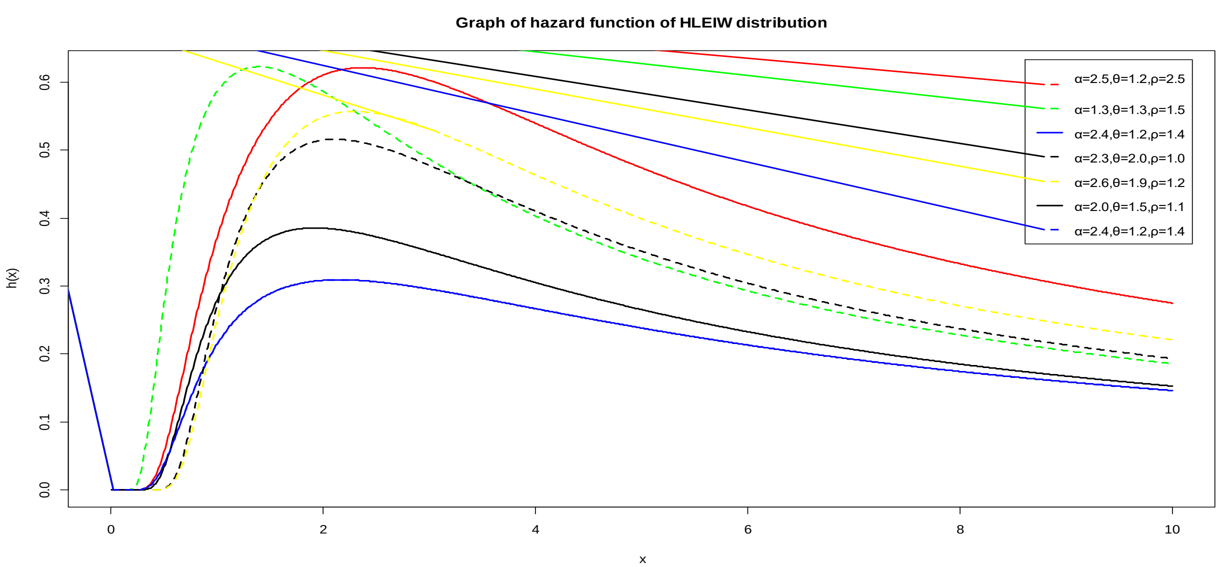

Figure 2.0. Graph of the hazard function of the $HLEIW$ distribution

- It could be observed from Figure 2.0 that the failure rate function of the $HLEIW$ model exhibits the form of inverted bathtub-shaped failure rate.

3.0 Important representation

In this subsection, an important tool for the expansion of the PDF and CDF for HLEIW is provided. From the generalized binomial series given by

$$(1+m)^{-a} = \sum_{i=0}^{\infty} (-1)^i \binom{a+i-1}{i} m^i \tag{9}$$

For $|m| < 1$ and a is a positive real non-integer. Then, by applying the binomial theorem (8) in (5), the density function of $HLEIW$ distribution becomes

$$f(x;\alpha,\theta,\rho) = 2\alpha\theta\rho \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{i+1}{i} \binom{\rho(i+1)-1}{j} x^{-\theta-1} e^{-(1+j)\alpha x^{-\theta}} \tag{10}$$

3.1 Statistical characteristics of $HLEIW$ distribution

3.1.1 The quantiles, median and the upper quartile

An expression for the quantile and the median of $HLEIW$ are obtained in this subsection.
The quantile x_q of the $HLEIW$ is represented as follows

$$x_q = \left(-ln\left[1-\left(1-\left(\frac{1-q}{1+q}\right)^{1/\rho}\right)^{1/\alpha}\right]\right)^{-1/\theta} \tag{11}$$

The median and the upper quartile of $HLEIW$ are found by putting $q = 0.5$ and 0.75 in (11), respectively, as follows:

$$x_{0.5} = \left(-ln\left[1-\left(1-\left(\frac{0.5}{1.5}\right)^{1/\rho}\right)^{1/\alpha}\right]\right)^{-1/\theta} \tag{12}$$

and

$$x_{0.75} = \left(-ln\left[1-\left(1-\left(\frac{0.25}{1.75}\right)^{1/\rho}\right)^{1/\alpha}\right]\right)^{1/\theta} \tag{13}$$

3.2. The v^{th} moment

If $X \sim HLEIW(\zeta)$, then the $\boldsymbol{v^{th}}$ moment of X can be obtained using

$$\mu'_v = E(X^v) = \int_0^{\infty} x^v f(x) dx. \tag{14}$$

By substituting from (10) in (14), we get the $\boldsymbol{v^{th}}$ moment as follows

$$\mu'_v = \alpha\rho \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{i+1}{i} \binom{\rho(i+1)-1}{j} \alpha^{\left({}^{v}/_{\theta}-1\right)} (j+1)^{-\left({}^{v}/_{\theta}-1\right)} \Gamma\left(1 - \frac{v}{\theta}\right). \tag{15}$$

By setting $v = 1$ in (15), we obtain the mean of X. Measures of skewness and kurtosis can also be calculated from the ordinary moments of X using the cumulants measure. The cumulant (κ_n) of X can be obtained as follows

$$(\kappa_n) = \mu'_n - \sum_{v=0}^{n-1} \binom{n-1}{v-1} \kappa_v (\mu'_{n-v}), \tag{16}$$

Where,

$$\kappa_1 = \mu'_1, \quad \kappa_1 = \mu'_2 - (\mu'_2)^2, \quad \kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + (\mu'_1)^3$$

And

$$\kappa_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3(\mu'_2)^2 + 12\mu'_2(\mu'_1)^2 - 6(\mu'_1)^4$$

Consequently, an expression for the skewness and the kurtosis can be calculated as follows $\kappa_s = {}^{\kappa_3}/_{(\sqrt{\kappa_2})^3}$ and $\kappa_u = {}^{\kappa_4}/_{(\kappa_2)^2}$ respectively.

3.3 Incomplete moment of $\boldsymbol{HLEIW}$ distribution

If $X \sim HLEIW(\zeta)$, then the $\boldsymbol{v^{th}}$ incomplete moment of X can be obtained using

$$\triangle^{(v)} = E(X^v) = \int_0^{t} x^v f(x) dx. \tag{17}$$

By substituting from (10) in (17), we get the $\boldsymbol{v^{th}}$ incomplete moment as follows

$$\triangle^{(v)} = \alpha\rho \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{i+1}{i} \binom{\rho(i+1)-1}{j} \alpha^{\left({}^{v}/_{\theta}-1\right)} (j+1)^{-\left({}^{v}/_{\theta}-1\right)} \Gamma\left(1 - \frac{v}{\theta}, \alpha(1+j)t^{-\theta}\right) \tag{18}$$

3.4. Moment generating function (mgf)

The mgf of $HLEIW(\zeta)$, say $M_X(t)$ is found using

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tX} f(x)dx = \sum_{v=0}^\infty \frac{t^v}{v!}\mu^{(v)} \tag{19}$$

Substituting (15) into (19), we obtain

$$M_X(t) = \alpha\rho \sum_{i,j,v=0}^\infty \frac{t^v}{v!}(-1)^{i+j}\binom{i+1}{i}\binom{\rho(i+1)-1}{j}\alpha^{\left({}^{v}/_{\theta}-1\right)}(j+1)^{-\left({}^{v}/_{\theta}-1\right)}\Gamma\left(1-\frac{v}{\theta}\right) \tag{20}$$

3.5 Entropy Function and δ −Entropy

The entropy function can be used to determine the level uncertainty related to X whose PDF $g(x)$. It plays a fundamental role in computer science, engineering, and others. The Renyi entropy of X, say $I_\varphi(X)$, is determined by

$$I_\varphi = \frac{1}{1-\varphi} log \int_{-\infty}^\infty g^\varphi(x)dx \tag{21}$$

If X ~ $HLEIW(\zeta)$, then $I_\varphi(X)$ is obtained by

$$I_\varphi = \frac{1}{1-\varphi} log\left(2^\varphi\alpha^\varphi\theta^{\varphi-1}\rho^\varphi(\alpha j+\varphi)^{1-{}^{(\theta+1)\varphi}/_{\theta}}M^*\Gamma\left(\frac{(\theta+1)(\varphi-1}{\theta}+1\right)\right) \tag{22}$$

where

$$M^* = \sum_{i,j=0}^\infty (-1)^{i+j}\binom{2\varphi+i-1}{i}\binom{(\rho-i)\varphi+\rho i}{j}$$

Consequently, the δ-entropy of X, say $Z_\varphi(X)$ is given by

$$Z_\delta(X) = \frac{1}{1-\varphi} log\left[1-(1-\varphi)I_\varphi(X)\right] \tag{23}$$

4.0 Maximum Likelihood Estimation (MLE)

Suppose $x_1, x_2, \dots, x_n$ are the observed sample values obtained from the $HLEIW$ distribution. The log-likelihood (l) function is defined as follows:

$$l(\alpha,\theta,\rho) = nlog(\alpha) + nlog(\theta) + nlog(\rho) - (\theta+1)\sum_{i=1}^{\infty} log(x_i) - \alpha \sum_{i=1}^{\infty} {x_i}^{-\theta}$$

$$+(\rho-1)\sum_{i=1}^{n} log\left[1-\left(e^{-{x_i}^{-\theta}}\right)^{\alpha}\right] - 2\sum_{i=1}^{n} log\left(1+\left[1-\left(e^{-x^{-\theta}}\right)^{\alpha}\right]^{\rho}\right) \quad (24)$$

Maximizing $l(\alpha,\theta,\rho)$ with respect to η, φ, and ρ, we obtain the following system of nonlinear equations:

$$\frac{n}{\alpha} + \sum_{i=1}^{\infty} {x_i}^{-\theta} + \sum_{i=1}^{n}\left(\frac{\alpha {x_i}^{-\theta-1}{P_i}^{x}}{(1-{P_i}^{x})}\right) - 2\rho\sum_{i=1}^{n}\left(\frac{\alpha x^{-\theta}[1-v]^{\rho}\left(e^{-x^{-\theta}}\right)^{\alpha}}{(1+[1-{P_i}^{x}]^{\rho})[1-{P_i}^{x}]}\right) = 0$$

$$\frac{n}{\theta} - \sum_{i=1}^{\infty} log(x_i) + \alpha\sum_{i=1}^{n}\left(\frac{\theta x^{-\theta} logx}{(x)}\right) + (\rho-1)\alpha\theta\sum_{i=1}^{n}\left(\frac{x^{-\theta}{P_i}^{x} logx}{x(1+[1-{P_i}^{x}]^{\rho})}\right)$$

$$+2\rho\theta\sum_{i=1}^{n}\left(\frac{\theta x^{-\theta} logx}{(1+[1-{P_i}^{x}])[1-{P_i}^{x}]}\right) = 0$$

$$\frac{n}{\rho} - \sum_{i=1}^{\infty} log[{P_i}^{x}] - 2\sum_{i=1}^{n}\left(\frac{([1-{P_i}^{x}]^{\rho})log[1-{P_i}^{x}]}{(1+[1-{P_i}^{x}]^{\rho})}\right) = 0$$

Where, ${P_i}^{x} = \left(e^{-x^{-\theta}}\right)^{\alpha}$

5.0 Applications

In this section, we compare the fit of the HLEIW model and some other competing models using one pig data set as reported by Bjerkedal (1960). We measure how well the HLEIW distribution performs compared to the half logistic Inverted Weibull (HLIW), Exponentiated Inverted Weibull (EIW), and Inverted Weibull (IW) distribution. For each model, we obtained the estimate of the parameters by using the maximum likelihood method and assessed the goodness-of-fit by using the following information criteria: Akaike information criterion (AIC), Consistent Akaike information criterion (CAIC), Hannan Quinin information criterion (HQIC), and Kolmogorov Smirnov (KS statistic. In general, the smaller the value of the information criteria, the better the model fit to the data.

The data used represent the survival times (in days) of 72 guinea pigs. The real data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli reported by Bjerkedal (1960). They are the Regiment 4.3, Study M.: 10, 33, 44, 56, 59, 72, 74, 77, 92, 93, 96, 100, 100, 102, 105, 107, 107, 108, 108, 108, 109, 112, 113, 115, 116, 120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176, 183, 195, 196, 197, 202, 213, 215, 216, 222, 230,231, 240, 245, 251, 253, 254, 255, 278, 293, 327, 342, 347, 361, 402, 432, 458, 555. The exploratory data analysis for the pig data set is given in Table 1, which shows that the data is over-dispersed, positively skewed and Leptokurtic with excess kurtosis of 1.9885. Figure 3.0 clearly indicates that the [pig data exhibits an increasing failure rate and skewed to the right.

The ML estimates (standard errors -SEs- in parentheses and confidence interval (CI) in curly brackets) as well as the AIC, CAIC, and KS statistics are given in Table 3. All four goodness-of-fit statistics shows that the $HLEIW$ model gives the best fit.

Table 1. Summary Statistic of the pig data

Range	Lower quartile	Median	Upper quartile	Mean	Variance	Skewness	Kurtosis
545	108	149.5	224.0	176.8	10705.1	1.341	4.988

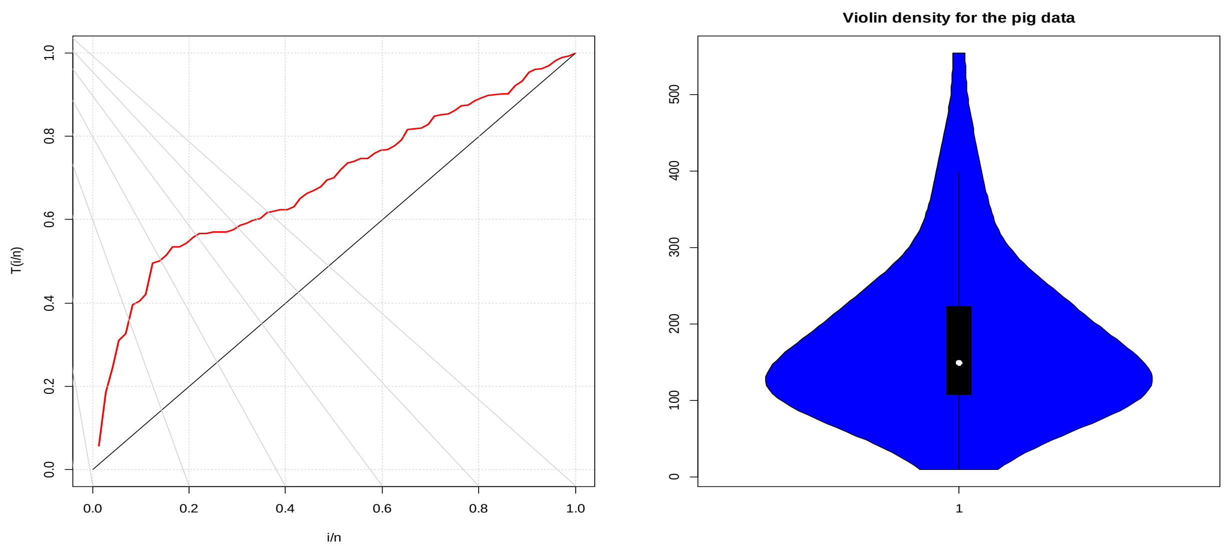

Figure 3.0 Total Time on Test (TTT) plot and Violin Plot for the pig data

Table 2.0 model fits for the pig data

$Model$	α	θ	ρ	$-l$	AIC	$CAIC$	KS
$HLEIW$	19.821 (3.194)	0.435 (0.042)	10.8907 (3.3157)	436.553	879.105	879.458	0.2288
$HLIW$	-	5.2897	0.0579	0.0112	1073.017	1073.19	0.5071
EIW	24.1408	0.7146	--	467.821	939.642	939.816	0.2741
IW	-	0.2692 (0.0262)	-	570.802	1143.603	1143.661	0.6713

From Table 2.0, it could be observed that the HLEIW model has the smallest AIC, CAIC, and KS statistics which is an indication that is the best model among all other models consider in this study for the modeling of pig's data.

6.0 Conclusion

A new three-parameter distribution called the $HLEIW$ distribution is developed. The characteristic of the $HLEIW$ distribution is that its failure rate function can be decreasing, increasing, bathtub-shaped and unimodal depending on its parameter values. Several statistical properties of the new distribution such as its probability density function, its cumulative density function, quantiles, moments, incomplete moments, moments generating functions, Renyi and ρ-entropies are derived. Fitting the $HLEIW$ model to a pig data set demonstrate the flexibility and usefulness of the proposed distribution because it provides a good fit when compared with other competing models considered in this study.

References

Anwar, M. and Bibi, A. (2018). The Half-Logistic Generalized Weibull Distribution. J. Probab. Stat., 12.

Bjerkedal, T. (1960). Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli. American Journal of Hygiene 72:130–148.

Bourguignon, M., Silva, R. B. and Cordeiro, G. M. (2014). The Weibull-G Family of Probability distributions. *Journal of Data Science,* 12: 53-68.

Fayomi, A. (2019). Type I half logistic power Lomax distribution: Statistical properties and application. Adv. Appl. Stat., 54, 85–98.

Flair, A., Elsalloukh, H. Mendi, E and Milanova, M. (2012). The exponentiated inverted Weibull Distribution, Appl. Math. Inf. Sci. 6, No. 2, 167-171.

Khan, M. S., Pasha, G. R. and Pasha, A. H. (2008). Theoretical analysis of inverse Weibull distribution, WSEAS Transactions on Mathematics, 7, 2.

Mudholka, G. S., Srivastava, D. K. and Freimer, M. (1993). The exponentiated Weibull family: a real analysis of the bus motor failure data.

Mudholkar, G. S. and Hutson, A. D. (1995) Exponentiated Weibull family: some properties and flood data application, Commun. Statistical Theory and Method. 25, 3050-3083

Ogunde, A. A., Oseghale, O. I., Olayode F., and Laoye V. E. (2022). Modified Extended Inverted Weibull Distribution with Application to Neck Cancer Data. Journal of Mathematics Research, Vol. 14 (2), 39-51.

Ogunde, A. A., Oseghale, O. I. and Audu, A. T. (2017). Performance Rating of the Type 1 Half Logistic Gompertz Distribution: An Analytical Approach, American Journal of Mathematics and Statistics, Vol. 7 (3), 93-98.

Ogunde, A. A. Fatoki, O., Omosigho D. O., Ajayi B. (2018). On Statistical Properties of the Weibull Inverted Weibull Distribution. Edited proceeding of the Second International conference of the Nigerian Statistical Society.

Ogunde, A. A., Subhankar, D. and Almetawally, E. M. (2024). Half logistic Generalized Distribution for modeling Hydrological Data. Annals of Data Science, Springer-Verlag GmbH Germany.

Zein-Eldin, R. A. Chesneau, C. Jamal, F. and Elgarhy, M. (2019). Different estimation methods of type I half-logistic Topp-Leone distribution. Mathematics, 7, 985.